7.8 Repeated Eigenvalues

We have our system

$$\vec{x}' = A\vec{x}$$

and suppose $r = \rho$ is a repeated root.

Recall
- algebraic multiplicity is the number of times an eigenvalue appears as a root in the characteristic equation.
- geometric multiplicity is the number of linearly independent eigenvector of an eigenvalue.

$$1 \leq \text{geo. mult.} \leq \text{alg. mult.}$$

We know what happens when geo. mult. = alg. mult.
We get m linearly independent eigenvectors,
so we get m linearly independent

Solutions.

What do we do if geo. mult. < alg. mult.?

Suppose $r = \rho$ is a double eigenvalue of A and $\vec{\xi}$ is the only eigenvector. Then we have the solution

$$\vec{x}^{(1)}(t) = \vec{\xi} e^{\rho t}$$

where

$$(A - \rho I)\vec{\xi} = 0$$

To find a second solution, let's try

$$\vec{x}^{(2)}(t) = \vec{\xi} t e^{\rho t}$$

Then
$$\vec{x}^{(2)\prime}(t) = \vec{\xi} e^{\rho t} + \vec{\xi} \rho t e^{\rho t}$$

and

$$A\vec{x}^{(2)}(t) = A\vec{\xi}\, t e^{\rho t}$$
$$= \rho\vec{\xi}\, t e^{\rho t}$$

Now,

$$\vec{\xi} e^{\rho t} + \vec{\xi}\rho t e^{\rho t} = \rho\vec{\xi}\, t e^{\rho t}$$

$$\vec{\xi} e^{\rho t} = 0$$

and so $\vec{\xi} = 0$ is the only solution for our guess of $\vec{x}^{(2)}(t)$. $\vec{x}^{(2)\prime}(t)$ has terms involving $te^{\rho t}$ and $e^{\rho t}$, so we modify our solution $\vec{x}^{(2)}(t)$ to be

$$\vec{x}^{(2)}(t) = \vec{\xi}\, t e^{\rho t} + \vec{\eta}\, e^{\rho t}$$

Now,

$$\vec{x}^{(2)\prime}(t) = \vec{\xi} e^{\rho t} + \vec{\xi} \rho t e^{\rho t} + \vec{\eta} \rho e^{\rho t}$$

and

$$A\vec{x}^{(2)}(t) = A\vec{\xi} t e^{\rho t} + A\vec{\eta} e^{\rho t}$$
$$= \rho \vec{\xi} t e^{\rho t} + A\vec{\eta} e^{\rho t}$$

Therefore, we must have

$$\vec{\xi} e^{\rho t} + \vec{\xi} \rho t e^{\rho t} + \vec{\eta} \rho e^{\rho t} = \rho \vec{\xi} t e^{\rho t} + A\vec{\eta} e^{\rho t}$$

$$\vec{\xi} e^{\rho t} = A\vec{\eta} e^{\rho t} - \vec{\eta} \rho e^{\rho t}$$

$$\vec{\xi} = A\vec{\eta} - \rho \vec{\eta}$$

$$\vec{\xi} = (A - \rho I)\vec{\eta}$$

Such a vector $\vec{\eta}$ is called a generalized eigenvector of A. Since $(A-\rho I)\vec{\xi} = 0$, we can alternatively use

$$(A-\rho I)^2 \vec{\eta} = 0 .$$

<u>Ex</u> $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$

The eigenvalues are

$$\det(A - rI) = 0$$

$$(1-r)(3-r)+1 = 0$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = 2 .$$

The eigenvector for $r=2$ is

$$(A-2I)\vec{\xi} = 0$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \vec{\xi} = 0$$

so $\vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the only eigenvector.

One solution is of the form

$$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

and the second solution is of the form

$$\vec{x}^{(2)}(t) = \vec{\xi} t e^{2t} + \vec{\eta} e^{2t}$$
$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \vec{\eta} e^{2t}$$

where $\vec{\eta}$ is a generalized eigenvector.

Now,

$$(A - 8I)\vec{\eta} = \vec{\xi}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$-\eta_1 - \eta_2 = 1$$

$$\eta_2 = -\eta_1 - 1$$

so

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ -\eta_1 - 1 \end{pmatrix} = \eta_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Therefore,

$$\vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + \cancel{\eta_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}} e^{2t}$$

since

$$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

Our Fundamental matrix is

$$\Psi(t) = \begin{pmatrix} | & | \\ \vec{x}^{(1)} & \vec{x}^{(2)} \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} & t e^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} 1 & t \\ -1 & -t-1 \end{pmatrix}$$

and so the Wronskian is

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}](t) = e^{4t}(-t-1+t)$$

$$= -e^{4t}$$

so $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ form a fundamental set of solutions.

7.9 Nonhomogeneous Linear Systems

Consider the system

$$\vec{x}\,' = P(t)\vec{x} + \vec{g}(t)$$

The general solution is of the form

$$\vec{x} = \underbrace{c_1\vec{x}^{(1)}(t) + \cdots + c_n\vec{x}^{(n)}(t)}_{\text{homogeneous soln.}} + \underbrace{\vec{v}(t)}_{\text{particular soln.}}$$

Diagonalization

Suppose A is diagonalizable, and consider the system

$$\vec{x}\,' = A\vec{x} + \vec{g}(t).$$

Then we have that $A = TDT^{-1}$ so

$$\vec{x}\,' = TDT^{-1}\vec{x} + \vec{g}(t)$$

$$T^{-1}\vec{x}\,' = DT^{-1}\vec{x} + T^{-1}\vec{g}(t)$$

Let $\vec{x} = T\vec{y}$ and $\vec{g}(t) = T\vec{h}(t)$. Now,

$\vec{x}' = T\vec{y}'$

$$\vec{y}' = D\vec{y} + \vec{h}(t)$$

which is a system of n uncoupled first-order linear ODEs,

$$y_j' = r_j y_j + h_j \quad , \quad j = 1, \dots, n$$

which we know has solution

$$y_j = \underbrace{e^{r_j t} \int_{t_0}^{t} e^{-r_j s} h_j(s)\, ds}_{\text{particular soln.}} + \underbrace{c_j e^{r_j t}}_{\text{homogeneous soln.}}, \quad j = 1, \dots, n.$$

Finally, $\vec{x} = T\vec{y}$ sends us back to original variables.

Undetermined Coefficients

Guess the form of the particular solution and determine coefficients.

Note P must be a constant matrix and components of $\vec{g}(t)$ must be "guessable" functions.

Variation of Parameters

Consider the system

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

with Ψ the fundamental matrix of

$$\vec{x}' = P(t)\vec{x}.$$

The general solution of the homogeneous system is $\vec{x} = \Psi\vec{c}$. We suppose that the nonhomogeneous system has a solution of the form

$$\vec{x} = \Psi(t)\vec{u}(t)$$

where $\vec{u}$ is to be determined. Now,

$$\vec{x}' = \Psi'\vec{u} + \Psi\vec{u}'$$

and so

$$\vec{x}' = P(t)\vec{x} + \vec{g}$$

$$\Psi'\vec{u} + \Psi\vec{u}' = P(t)\Psi\vec{u} + \vec{g}$$

Now, $\Psi' = P\Psi$, so

$$\Psi\vec{u}' = \vec{g}$$

therefore

$$\vec{u}' = \Psi^{-1}\vec{g}$$

We get

$$\vec{u} = \int \Psi^{-1} \vec{g} \, dt + \vec{c}$$

and so

$$\begin{aligned} \vec{x} &= \Psi \vec{u} \\ &= \underbrace{\Psi \vec{c}}_{\text{homogeneous soln.}} + \underbrace{\Psi \int_{t_0}^{t} \Psi^{-1}(s) \vec{g}(s) \, ds}_{\text{particular soln.}} \end{aligned}$$

If we have the IC $\vec{x}(t_0) = \vec{x}^0$,

then $$\vec{x}(t_0) = \Psi(t_0) \vec{c} = \vec{x}^0$$

$$\vec{c} = \Psi^{-1}(t_0) \vec{x}^0$$

and so

$$\vec{x}(t) = \Psi(t) \Psi^{-1}(t_0) \vec{x}^0 + \Psi(t) \int_{t_0}^{t} \Psi^{-1}(s) \vec{g}(s) \, ds.$$

Laplace Transforms

Transform of a vector is computed component by component.

$$\mathcal{L}\{\vec{x}(t)\} = \vec{X}(s)$$

<u>Ex</u> $\vec{x}' = A\vec{x} + \vec{g}(t)$

$$= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 3e^{-t} \\ 3t \end{pmatrix}$$

$$\mathcal{L}\{\vec{x}'\} = s\vec{X}(s) - \vec{x}(0)$$

$$\mathcal{L}\{A\vec{x} + \vec{g}(t)\} = A\vec{X}(s) + \begin{pmatrix} \frac{3}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

So

$$s\vec{X}(s) - \vec{x}(0) = A\vec{X}(s) + \begin{pmatrix} \frac{3}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

$$s\vec{X}(s) - A\vec{X}(s) = \vec{x}(0) + \begin{pmatrix} \frac{3}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

$$(sI - A)\vec{X}(s) = \vec{x}(0) + \begin{pmatrix} \frac{3}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

$$\vec{X}(s) = (sI - A)^{-1}\left(\vec{x}(0) + \begin{pmatrix} \frac{3}{s+1} \\ \frac{3}{s^2} \end{pmatrix}\right)$$